

# Classification of complete Finsler manifolds through a second order differential equation

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## Abstract

By using a certain second order differential equation, the notion of adapted coordinates on Finsler manifolds is defined and some classifications of complete Finsler manifolds are found. Some examples of Finsler metrics, with positive constant sectional curvature, not necessarily of Randers type nor projectively flat, are found. This work generalizes some results in Riemannian geometry and open up, a vast area of research on Finsler geometry.

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## Introduction

Differential equations play an essential role in the study of global differential geometry, particularly the second order differential equation  $\nabla \nabla \rho = \phi g$ , which appears often in the study of a Riemannian manifold  $(M, g)$ . In the above equation,  $\rho$  is the divergence of a conformal Killing vector field,  $\phi$  is a  $C^\infty$  function on  $M$  and  $\nabla$  is the covariant derivative of Levi-Civita connection. This differential equation has proved to be very fruitful and has been studied by many authors not to be mentioned here, see for instance [10], [17], [20], [21] and [22]. Geometrically, the existence of a solution to this differential equation is equivalent to the existence of a certain conformal transformation, which takes geodesic circles into geodesic circles - a geodesic circle being a curve with constant curvature and zero torsion and is a generalization of circle in the Euclidean space - see for example [11] and [21]. In Physics, this differential equation is closely connected to the study of collineations

in General Relativity [15]. The above differential equation appears also in the study of pseudo-Riemannian manifolds, see for example [8] and [14].

In Finsler geometry, this differential equation has been investigated in [1] and [2], using a method of calculus of variation. The results there obtained may be considered as a very special case of the main theorem presented in this work.

We propose to consider some possible applications of this differential equation in a Finslerian setting, having in mind the following remarks: First of all existence of a solution permits the definition of a new adapted coordinate system, which will somehow play the same role in Finsler geometry try as the normal coordinate system in Riemannian geometry (§1, §2 and §3) in which it has proved to be a powerful tool, while its usefulness has been limited in the Finsler geometry. In fact, in the latter case the exponential map is only  $C^1$  at the zero section of  $TM$ , while it is  $C^\infty$  in the former case [1], [3]. Next, the following classification theorem can be proved (§5).

**Theorem:** *Let  $(M, g)$  be a connected complete Finsler manifold of dimension  $n \geq 2$ . If  $M$  admits a non-trivial solution of  $\nabla^H \nabla^H \rho = \phi g$ , where  $\nabla^H$  is the Cartan horizontal covariant derivative, then depending on the number of critical points of  $\rho$ , i.e. zero, one or two respectively, it is conformal to*

(a) *A direct product  $J \times \overline{M}$  of an open interval  $J$  of the real line and an  $(n - 1)$ -dimensional complete Finsler manifold  $\overline{M}$ .*

(b) *An  $n$ -dimensional Euclidean space.*

(c) *An  $n$ -dimensional unit sphere in an Euclidean space.*

It should be remarked that the role played by Cartan derivative in the above theorem is essential and can not be replaced by, for example the Berwald derivative. Well known examples of Finsler metrics with positive constant curvature are either of Randers type or are projectively flat [3],[4],[5],[7],[9]. As yet another next application of this differential equation, we find some examples of Finsler metrics with positive constant sectional curvature which are not necessarily of Randers type nor projectively flat (§4). More precisely, by using a Finsler metric with positive constant sectional curvature together with the above theorem, one can construct Finsler metrics of positive constant curvature in higher dimensions (§6).

## Preliminaries.

Let  $M$  be a real  $n$ -dimensional manifold of class  $C^\infty$ . We denote by  $TM \rightarrow M$  the bundle of tangent vectors and by  $\pi : TM_0 \rightarrow M$  the fiber bundle of non-zero tangent vectors. A *Finsler structure* on  $M$  is a function  $F : TM \rightarrow [0, \infty)$ , with the following properties: (I)  $F$  is differentiable ( $C^\infty$ ) on  $TM_0$ ; (II)  $F$  is positively homogeneous of degree one in  $y$ , i.e.  $F(x, \lambda y) = \lambda F(x, y)$ ,  $\forall \lambda > 0$ , where  $(x, y)$  is an element of  $TM$ . (III) The Hessian matrix of  $F^2$ ,  $(g_{ij}) := \left( \frac{1}{2} \left[ \frac{\partial^2}{\partial y^i \partial y^j} F^2 \right] \right)$ , is positive definite on  $TM_0$ . A *Finsler manifold* is a pair consisting of a differentiable manifold  $M$  and a Finsler structure  $F$  on  $M$ . The tensor field  $g$  with the components  $g_{ij}$  is called the *Finsler metric tensor*. Hereafter, we denote a Finsler manifold by  $(M, g)$ . Let  $V_v TM = \ker \pi_*$  be the set of vectors tangent to the fiber through  $v \in TM_0$ . Then a *vertical vector bundle* on  $M$  is defined by  $VTM := \bigcup_{v \in TM_0} V_v TM$ . A *non-linear connection*

on  $TM_0$  is a complementary distribution  $HTM$  for  $VTM$  on  $TTM_0$ . Therefore we have the decomposition  $TTM_0 = VTM \oplus HTM$ . The pair  $(HTM, \nabla)$ , where  $HTM$  is a non-linear connection on  $TM$  and  $\nabla$  a linear connection on  $VTM$ , is called a *Finsler connection* on the manifold  $M$ . Using the local coordinates  $(x^i, y^i)$  on  $TM$ , called the line elements, we have the local field of frames  $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}\}$  on  $TTM$ . Given a non-linear connection one can choose a local field of frames  $\{\frac{\delta}{\delta x_i}, \frac{\partial}{\partial y_i}\}$  adapted to the above decomposition i.e.  $\frac{\delta}{\delta x_i} \in \Gamma(HTM)$  and  $\frac{\partial}{\partial y_i} \in \Gamma(VTM)$ , the set of vector fields on  $HTM$  and  $VTM$  respectively. Here  $\frac{\delta}{\delta x_i} := \frac{\partial}{\partial x_i} - G_i^j \frac{\partial}{\partial y_j}$ , and  $G_i^j(x, y)$  are coefficients of the non-linear connection. A *Cartan connection* is a metric Finsler connection and its coefficients are defined by  $\Gamma_{jk}^{*i} = \frac{1}{2}g^{ih}(\frac{\delta g_{hk}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^k} - \frac{\delta g_{jk}}{\delta x^h})$ . Also we can write  $\Gamma_{jk}^{*i} = \gamma_{jk}^i - C_{jr}^i G_k^r + C_{jk}^r G_r^i - C_{kr}^i G_j^r$ , where  $\gamma_{jk}^i$  are *formal Christoffel symbols* of the second kind given by  $\gamma_{jk}^i = \frac{1}{2}g^{ih}(\frac{\partial g_{hk}}{\partial x^j} + \frac{\partial g_{hj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^h})$ . Let  $\overline{M}$  and  $M$  be two differentiable manifolds of dimension  $m$  and  $m+n$  respectively and let  $(u^\alpha)$  and  $(x^i)$  be local coordinate systems on them. We denote by  $(u^\alpha, v^\alpha)$  and  $(x^i, y^i)$  pairs of position and direction of the line elements of  $T\overline{M}$  and  $TM$ , where  $\alpha, \beta, \dots$  and  $i, j, \dots$  run over the range  $1, \dots, m$  and  $1, \dots, m+n$  respectively. Let  $f : \overline{M} \rightarrow M$  be a smooth map, given by  $(u^1, \dots, u^m) \rightarrow x^i(u^1, \dots, u^m)$ . The differential mapping of  $f$  is

$$\begin{aligned} f_* : T_u \overline{M} &\rightarrow T_x M, \\ (u^\alpha, v^\alpha) &\rightarrow (x^i(u), y^i(u, v)), \end{aligned}$$

where  $y^i(u, v) = B_\alpha^i v^\alpha$  and  $B_\alpha^i = \frac{\partial x^i}{\partial u^\alpha}$ . If  $f_*$  is injective at every point  $u$  of  $\overline{M}$ , that is, if  $\text{rank } [B_\alpha^i] = m$ , then  $\overline{M}$  is called an *immersed submanifold* or simply a *submanifold* of  $M$ . Next, consider an  $(m+n)$ -dimensional Finsler manifold  $(M, g)$ . The Finsler structure  $F$  induces on  $T\overline{M}$  a Finsler structure  $\overline{F}$  defined by  $\overline{F}(u, v) := F(x(u), y(u, v))$ . Putting  $\overline{g}_{\alpha\beta} := \frac{1}{2} \frac{\partial^2 \overline{F}^2}{\partial v^\alpha \partial v^\beta}$ , one obtains by direct calculation  $\overline{g}_{\alpha\beta}(u, v) = g_{ij}(x(u), y(u, v)) B_{\alpha\beta}^{ij}$ , where  $B_{\alpha\beta}^{ij} = B_\alpha^i B_\beta^j$ . Therefore the pair  $(\overline{M}, \overline{g})$  is a Finsler manifold, called *Finsler submanifold* of  $(M, g)$ .

A diffeomorphism  $f : (M, g) \rightarrow (N, h)$  between  $n$ -dimensional Finsler manifolds  $(M, g)$  and  $(N, h)$  is called *conformal* if each  $(f_*)_p$  for  $p \in M$  is angle-preserving, and in this case two Finsler manifolds are called *conformally equivalent* or simply *conformal*. If  $M = N$  then  $f$  is called a *conformal transformation*. It can be easily checked that a diffeomorphism  $f$  is conformal if and only if,  $f^*h = \rho g$  for some positive function  $\rho : M \rightarrow \mathbb{R}^+$ . The diffeomorphism  $f$  is called an *homothety* if  $\rho = \text{constant}$  and an *isometry* if  $\rho = 1$ . Now let's consider two Finsler manifolds  $(M, g)$  and  $(M, \overline{g})$ , then these two manifolds are conformal if and only if,  $\overline{g} = \rho(x) g$ .

Throughout this paper, all manifolds are supposed to be connected.

# 1 Finsler manifolds admitting a non-trivial solution of $\nabla^H \nabla^H \rho = \phi g$ .

Let  $(M, g)$  be an  $n$ -dimensional Finsler manifold and  $\rho : M \rightarrow [0, \infty)$  a scalar function on  $M$  given by the following second order differential equation

$$\nabla^H \nabla^H \rho = \phi g, \quad (1.1)$$

where  $\nabla^H$  is the Cartan horizontal covariant derivative and  $\phi$  is a function of  $x$  alone, then we say that the Eq. (1.1) has a solution  $\rho$ .<sup>1</sup>

In this section we consider the non-trivial (i.e. non-constant) solution  $\rho$  of the Eq.(1.1). The connected component of a regular hypersurface defined by  $\rho = \text{constant}$ , is called a *level set of  $\rho$* . Let's denote by  $\mathbf{grad}\rho$  the gradient vector field of  $\rho$  which is locally written in the form  $\mathbf{grad}\rho = \rho^i \frac{\partial}{\partial x^i}$ , where  $\rho^i = g^{ij} \rho_j$  and  $i, j, \dots$  run over the range  $1, \dots, n$ . Contracting (1.1) with  $\rho^k$ , we get  $\rho^k (\nabla_k^H \rho^l) = \phi \rho^l$  or equivalently  $\rho^k \frac{\partial \rho^l}{\partial x^k} + \Gamma_{jk}^{*l} \rho^j \rho^k = \phi \rho^l$ , which shows that the trajectories of the vector field  $\mathbf{grad}\rho$  are geodesic arcs.

Therefore we can choose a local coordinates  $(u^1 = t, u^2, \dots, u^n)$  on  $M$  such that  $t$  is parameter of the geodesic containing a trajectory of the vector field  $\mathbf{grad}\rho$  and the level sets of  $\rho$  are defined by  $t = \text{constant}$ , called respectively the  *$t$ -geodesic* and the  *$t$ -levels* of  $\rho$ . In the local coordinates  $(t, u^2, \dots, u^n)$ ,  $t$ -levels of  $\rho$  are defined by  $t = \text{constant}$ , so  $\rho$  may be considered as a function of  $t$  alone. In the sequel we will refer to this coordinates as an *adapted coordinates*.

The differential equation of  $t$ -geodesics are given by

$$\frac{d}{dt} \frac{du^i}{dt} + 2G^i = \varrho \frac{du^i}{dt}, \quad (1.2)$$

where  $\varrho$  is a scalar function of  $t$  and  $G^i = \frac{1}{2} \Gamma_{jk}^{*i} \frac{du^j}{dt} \frac{du^k}{dt}$  are spray coefficients. Since  $\frac{du^i}{dt} = \delta_1^i$ , where  $\delta_j^i$  is the Kronecker symbol, from the above geodesic differential equation we have

$$G^1 = \frac{1}{2} \varrho \quad \text{and} \quad G^\alpha = 0, \alpha = 2, 3, \dots, n \quad (1.3)$$

**Proposition 1.** *Let  $(M, g)$  be a Finsler manifold admitting a non-trivial solution of (1.1). Then  $g_{1\beta} = g_{\beta 1} = 0$ , where  $\beta = 2, 3, \dots, n$  and  $g_{11} = 1$ .*

*Proof.* The  $t$ -geodesics are normal to the  $t$ -levels of  $\rho$ , so we have

$$g_{1\beta} = g_{\beta 1} = 0, \quad \beta = 2, 3, \dots, n. \quad (1.4)$$

Putting (1.4) in the definition of  $\Gamma_{11}^{*i}$  and using (1.3) we get

$$\frac{1}{2} g^{i1} \delta_1 g_{11} - \frac{1}{2} g^{i\alpha} \delta_\alpha g_{11} = 0. \quad (1.5)$$

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<sup>1</sup>In the local coordinate system, the Eq. (1.1) is given by  $\nabla_k^H \rho_l = \frac{\partial}{\partial x^k} \rho_l + \Gamma_{lk}^{*j} \rho_j = \phi g_{kl}$ , where  $\rho_l = \frac{\partial \rho}{\partial x^l}$ .

By replacing the index  $i$  by  $\beta$  in the above equation, we get  $\delta_\alpha g_{11} = 0$ . As a consequence of (1.3), the equation  $\frac{\delta}{\delta x_i} = \frac{\partial}{\partial x_i} - G_i^j \frac{\partial}{\partial y_j}$  becomes  $\frac{\delta}{\delta u^j} = \frac{\partial}{\partial u^j}$ , therefore we have  $\partial_\alpha g_{11} = 0$ . Hence by a suitable choice of  $t$ , we can assume

$$g_{11} = 1. \quad (1.6)$$

□

**Proposition 2.** *Let  $(M, g)$  be a Finsler manifold admitting a non-trivial solution of (1.1). Then the spray coefficients vanish.*

*Proof.* We have  $G^1 = \frac{1}{2} \Gamma_{ij}^{*1} \frac{du^i}{dt} \frac{du^j}{dt}$ , therefore from (1.4) and (1.6), we get  $G^1 = 0$  and then from (1.3), we have

$$G^i = 0. \quad (1.7)$$

□

As a consequence of the above proposition,  $t$  may be regarded as the arc-length parameter of  $t$ -geodesics.

**Theorem 1.** *Let  $(M, g)$  be an  $n$ -dimensional Finsler manifold admitting a non-trivial solution of (1.1), then  $(M, g)$  is a projectively flat Finsler manifold or is a direct product  $I \times \overline{M}$  of an open interval  $I$  of the real line and an  $(n-1)$ -dimensional Finsler manifold  $\overline{M}$ .*

*Proof.* Let's consider the local coordinates  $(t, u^2, \dots, u^n)$  on  $M$ , where  $t$  is the arc-length parameter. Then the geodesic equation of  $(M, g)$  becomes  $\frac{d^2 u^i}{dt^2} + \Gamma_{jk}^{*i} \frac{du^j}{dt} \frac{du^k}{dt} = 0$  and from Proposition (2), we get  $\frac{d^2 u^i}{dt^2} = 0$ . If all geodesics of  $(M, g)$  are parameterized by  $t$  as arc-length then they are straight lines and by definition  $(M, g)$  is a projectively flat Finsler manifold. If not, a number of geodesics of  $(M, g)$  should be parameterized by  $t$  and others by  $(u^\alpha)$ , then they will lie respectively on a straight line and an hypersurface which is a  $t$ -level of  $\rho$ . Therefore  $(M, g)$  is a direct product  $I \times \overline{M}$ , where  $I$  is a real line and  $\overline{M}$  is an  $(n-1)$ -dimensional Finsler manifold, diffeomorphic to  $t$ -levels of  $\rho$ . □

From (1.4) and (1.6), in local coordinates  $(u^1 = t, u^2, \dots, u^n)$ , the components of the Finsler metric tensor  $g$  is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \dots & 0 \\ 0 & g_{22} \dots g_{2n} \\ \vdots & \dots \\ 0 & g_{n2} \dots g_{nn} \end{pmatrix}.$$

**Lemma 1.** *Let  $(M, g)$  be a Finsler manifold and  $\rho$  a non-trivial solution of (1.1) on  $M$ . Then the Finsler metric form of  $M$  is given by*

$$ds^2 = (du^1)^2 + \rho'^2 f_{\gamma\beta} du^\gamma du^\beta, \quad (1.8)$$

where  $f_{\gamma\beta}$  is a Finsler metric tensor on a  $t$ -level of  $\rho$ .

*Proof.* Let  $\overline{M}$  be a  $t$ -level of  $\rho$ . Then the unit vector field  $\mathbf{i} = \frac{\text{grad}\rho}{\|\text{grad}\rho\|}$ , where  $\|\cdot\| = \sqrt{g(\cdot, \cdot)}$ , is normal to  $\overline{M}$  at any point of  $\overline{M}$  and the induced metric tensor  $\overline{g}_{\gamma\beta}$  of  $\overline{M}$  is given by  $\overline{g}_{\gamma\beta} = g_{ij}B_\gamma^i B_\beta^j$  where  $B_\gamma^i = \partial_\gamma u^i = \delta_\gamma^i$  (see [6]). Therefore we have

$$\overline{g}_{\alpha\beta} = g_{\alpha\beta}. \quad (1.9)$$

The  $h$ -second fundamental form of  $\overline{M}$  is defined by  $h_{\gamma\beta} := (\nabla_\gamma^H B_\beta^k) \mathbf{i}_k$  and

$$\partial_\beta B_\gamma^k + \Gamma_{ij}^{*k} B_\gamma^i B_\beta^j - \overline{\Gamma}_{\gamma\beta}^{*\alpha} B_\alpha^k = h_{\gamma\beta} \mathbf{i}^k, \quad (1.10)$$

where  $\overline{\Gamma}_{\gamma\beta}^{*\alpha}$  are Finsler connection's coefficients in  $(\overline{M}, \overline{g}_{\gamma\beta})$ . On the other hand,  $B_\beta^k \mathbf{i}_k = 0$ , so we have

$$h_{\gamma\beta} = -(\nabla_j^H \mathbf{i}_k) B_\gamma^j B_\beta^k. \quad (1.11)$$

Since the components of the unit vector field  $\mathbf{i}$  are  $\mathbf{i}^k = \delta_1^k$ , the equation (1.10) for  $k = 1$  reduces to

$$\Gamma_{\gamma\beta}^{*1} = h_{\gamma\beta}. \quad (1.12)$$

By  $h$ -covariant derivative of  $\rho_l = \mathbf{i}_l \|\text{grad}\rho\|$  and using (1.1), we have

$$(\nabla_k^H \|\text{grad}\rho\|) \mathbf{i}_l + \|\text{grad}\rho\| (\nabla_k^H \mathbf{i}_l) = \phi g_{lk}. \quad (1.13)$$

Contracting with  $\mathbf{i}^l$ , we have  $\nabla_k^H \|\text{grad}\rho\| = \phi \mathbf{i}_k$ , and by replacing in (1.13), we get

$$\nabla_k^H \mathbf{i}_l = \frac{\phi}{\|\text{grad}\rho\|} (g_{lk} - \mathbf{i}_k \mathbf{i}_l). \quad (1.14)$$

Substituting (1.14) into (1.11), we have

$$h_{\gamma\beta} = h g_{\gamma\beta}, \quad \text{where } h = \frac{-\phi}{\|\text{grad}\rho\|}. \quad (1.15)$$

Using  $\rho^l = g^{il} \rho_i$  and the fact that  $\rho$  is a function of  $t$  alone, we have  $\|\text{grad}\rho\| = \rho' \neq 0$ , where prime denotes the ordinary differentiation with respect to  $t$ . In the same way we get

$$\|\text{grad}\rho\| = \rho', \quad \phi = \rho'', \quad h = \frac{-\phi}{\|\text{grad}\rho\|} = \frac{-\rho''}{\rho'}. \quad (1.16)$$

By using (1.15) and replacing  $h$  in (1.12), we have  $\frac{\partial g_{\gamma\beta}}{\partial t} = \frac{2\rho''}{\rho'} g_{\gamma\beta}$ . Therefore the components  $g_{\gamma\beta}$  are written in the form

$$g_{\gamma\beta} = \rho'^2 f_{\gamma\beta}, \quad (1.17)$$

where  $f_{\gamma\beta}$  are functions of the  $2(n-1)$  coordinates  $(u^\alpha, v^\alpha)$ . Since the metric tensor  $g_{\gamma\beta}$  is positive definite, so is the matrix  $(f_{\gamma\beta})$ . Thus  $(f_{\gamma\beta})$  can be regarded as components of a Finsler metric tensor on  $\overline{M}$ .  $\square$

## 2 Curvature tensor of Cartan connection in adapted coordinates.

Let  $(M, g)$  be a Finsler manifold admitting a non-trivial solution of Eq.(1 .1), we want to compute the components of Cartan connection and its  $h$ -curvature tensor in terms of the adapted coordinates  $(t, u^2, \dots, u^n)$ .

The coefficients  $\rho'^2$  in (1 .17) are positive constants in every  $t$ -levels of  $\rho$ . Therefore if  $(N, f_{\gamma\beta})$  is an  $(n - 1)$ -dimensional Finsler manifold diffeomorphic to a  $t$ -level  $(\overline{M}, \overline{g}_{\gamma\beta})$ , then from (1 .9), the Finsler manifold  $(N, f_{\gamma\beta})$  and  $t$ -levels neighboring  $(\overline{M}, \overline{g}_{\gamma\beta})$  are locally homothetically diffeomorphic to each other. Indeed, the connection coefficients constructed from  $f_{\gamma\beta}$  on  $N$  have the same expression as the connection coefficients  $\overline{\Gamma}_{\beta\gamma}^{\alpha}$  constructed from the induced metric  $\overline{g}_{\gamma\beta}$  in  $\overline{M}$ . Therefore  $\Gamma_{ij}^{*k}$ , the components of Cartan connection on  $(M, g)$ , are given by

$$\begin{aligned}\Gamma_{11}^{*1} &= \Gamma_{1\beta}^{*1} = \Gamma_{\beta 1}^{*1} = \Gamma_{11}^{*\alpha} = 0, \\ \Gamma_{\gamma\beta}^{*1} &= -\frac{\rho''}{\rho'} g_{\gamma\beta} = -\rho' \rho'' f_{\gamma\beta}, \\ \Gamma_{1\beta}^{*\alpha} &= \Gamma_{\beta 1}^{*\alpha} = \frac{\rho''}{\rho'} \delta_{\beta}^{\alpha}, \\ \Gamma_{\alpha\beta}^{*\gamma} &= \overline{\Gamma}_{\alpha\beta}^{*\gamma},\end{aligned}\tag{2 .18}$$

where the last one comes from (1 .10), by replacing the index  $k$  with  $\alpha$ . The components of  $h$ -curvature tensor of Cartan connection is given by  $R_{hjk}^i = K_{hjk}^i + C_{hr}^i R_{jk}^r$  where  $K_{hjk}^i = \frac{\delta}{\delta u^k} \Gamma_{hj}^{*i} + \Gamma_{hj}^{*r} \Gamma_{rk}^{*i} - (j, k)$ ,  $R_{jk}^i = \frac{\delta}{\delta k} G_j^i - (j, k)$  and  $(j, k)$  denotes the interchange of indices  $j$  and  $k$ . Here we have  $R_{jk}^i = 0$  and so  $R_{hjk}^i = K_{hjk}^i$ . Therefore  $R_{hjk}^i$  of  $(M, g_{ij})$ , the components of the  $h$ -curvature tensor, are given by

$$\begin{aligned}R_{1\gamma 1}^{\alpha} &= -R_{\gamma 1 1}^{\alpha} = \left(\frac{\rho'''}{\rho'}\right) \delta_{\gamma}^{\alpha}, \\ R_{1\gamma\beta}^1 &= -R_{\gamma 1\beta}^1 = -\rho' \rho''' f_{\gamma\beta}, \\ R_{\delta\gamma\beta}^{\alpha} &= \overline{R}_{\delta\gamma\beta}^{\alpha} - (\rho'')^2 (f_{\gamma\beta} \delta_{\delta}^{\alpha} - f_{\delta\beta} \delta_{\gamma}^{\alpha}).\end{aligned}\tag{2 .19}$$

## 3 Critical points and their effects.

Let  $(M, g)$  be a Finsler manifold with Cartan connection and  $\rho$  a solution of the differential equation (1 .1) on  $M$ . If  $g(\text{grad}\rho, \text{grad}\rho) = 0$  in some points of  $M$ , then  $M$  possesses some interesting properties.

**Definition 1.** A point  $o$  of  $(M, g)$  is called a critical point of  $\rho$  if the vector field  $\text{grad}\rho$  vanishes at  $o$ , equivalently if  $\rho'(o) = 0$ .

Let  $\rho$  have a critical point  $o$ . We denote the distance function from  $o$  to an arbitrary point  $p \in M$  by a differentiable function  $d(o, p) := \inf L(\gamma)$ , where  $L(\gamma) =$

$\int_a^b F(\gamma, \frac{d\gamma}{dt})dt$  and  $\gamma : [a, b] \rightarrow M$  is a piecewise  $C^\infty$  curve with velocity  $\frac{d\gamma}{dt} \in T_{\gamma(t)}M$  such that  $\gamma(a) = o$  and  $\gamma(b) = p$ .

We recall that, along any  $t$ -geodesic, Eq.(1 .1) reduces to an ordinary differential equation

$$\frac{d^2\rho}{dt^2} = \phi(\rho), \quad (3 .20)$$

where  $\phi$  is a function of  $\rho$  which is differentiable at non-critical points.

**Proposition 3.** *Let  $(M, g)$  be a Finsler manifold and  $\rho$  a solution of Eq.(1 .1) on  $M$  with a critical point  $o$ . If one of the  $t$ -geodesics passes through  $o$ , then so do all of them.*

*Proof.* Let  $\overline{M}$  be a  $t$ -level of  $\rho$ , that is to say  $\rho$  and  $\rho'$  are constant on  $\overline{M}$  and there is no critical point on  $\overline{M}$ . Let  $p, q \in \overline{M}$  and denote by  $\ell(p)$  and  $\ell(q)$  the  $t$ -geodesics through  $p$  and  $q$ , respectively. The solution of Eq.(3 .20) is given by the same function  $\rho(t)$  on  $\ell(p)$  and  $\ell(q)$ , by uniqueness of solution of ordinary differential equations. Hence if one of the  $t$ -geodesics passes through a critical point at  $t_0 = 0$ , that is  $\rho'(t_0) = 0$ , then so do all of them.  $\square$

Moreover, from the uniqueness of solution of ordinary differential equations (3 .20), if we denote a point at a distance  $t$  from  $o$  on  $\ell(p)$  by  $p(t)$  and on  $\ell(q)$  by  $q(t)$ , then the points  $p(t)$  and  $q(t)$ , corresponding to the same value of  $t$ , are on the same  $t$ -level of  $\rho$ .

**Proposition 4.** *Let  $(M, g)$  be a complete Finsler manifold and  $\rho$  a solution of Eq. (1 .1) on  $M$  with a critical point  $o$ . Then  $o$  is an isolated point and  $t$ -levels of  $\rho$  are hyperspheres with center  $o$ .*

*Proof.* Let  $U(o)$  be a geodesically connected neighborhood of  $o$  and  $\overline{U}(o)$  be the closure of  $U(o)$ . If we denote by  $\overline{M}(p)$  a  $t$ -level of  $\rho$  through a point  $p$ , then  $\overline{M}(p)$  is closed, so is the intersection  $\overline{M}(p) \cap \overline{U}(o)$ , which contains its limit points. On the other hand, there is no critical point on  $\overline{M}(p)$ , so  $o$  is not a limit point of  $\overline{M}(p)$ . Therefore the distance function  $d(o, p)$  between  $o$  and the points of  $\overline{M}(p)$  has a minimum at an interior point of  $U(o)$ . Let  $p_1$  be this interior point and  $t_1$  the minimum value. Thus the  $t$ -geodesic which joins  $o$  to  $p$  is  $\ell(p_1)$ . We denote the geodesic hypersphere with center  $o$  and radius  $t_1$  by  $S^{n-1}(o, t_1)$ . Since  $\overline{M}(p) \cap S^{n-1}(o, t_1)$  is both closed and open in  $\overline{M}(p)$ , by taking into account the connectedness of  $\overline{M}(p)$ , it becomes evident that the geodesic hypersphere  $S^{n-1}(o, t_1)$  coincides with the  $t$ -level  $\overline{M}(p)$  and the proposition is proved.  $\square$

As a consequence of the above proposition, one can show easily that the number of critical points of  $\rho$  is not more than two.



## 4 Spaces of constant curvature.

Let  $P(v, X) \subset T_u(M)$  be a 2-plane generated by  $v$  and  $X \in T_u(M)$ , where  $(u, v)$  is the line element of  $TM$ . The *sectional curvature* with respect to  $P$  is given by

$$K(u, v, X) := \frac{g(R(X, v)v, X)}{g(X, X)g(v, v) - g(X, v)^2}.$$

If  $K$  is independent of  $X$ , then  $(M, g)$  is called *space of scalar curvature*. If  $K$  has no dependence on  $u$  or  $v$ , then the Finsler manifold is said to have *constant curvature*, see [1] or [3].

**Lemma 2.** *Let  $(M, g)$  be an  $n$ -dimensional Finsler manifold which admits a solution  $\rho$  of Eq. (1.1) with one critical point. Then the  $(n-1)$ -dimensional manifold  $\overline{M}$  with Finsler metric form  $\overline{ds}^2 = f_{\gamma\beta} du^\gamma du^\beta$  as defined in (1.17) is a space of positive constant curvature.*

*Proof.* Let  $o$  be a critical point of  $\rho$ . Proposition 4 implies that, there is a geodesically connected neighborhood  $U(o)$  of  $o$ , for which only the point  $o$  is critical. Hence from Proposition 1, the Finsler metric form in  $U(o)$  becomes

$$ds^2 = (du^1)^2 + \rho'^2 \overline{ds}^2,$$

where  $\overline{ds}^2 = f_{\gamma\beta} du^\gamma du^\beta$  is the Finsler metric form of an  $(n-1)$ -dimensional manifold  $\overline{M}$  diffeomorphic to the  $t$ -levels of  $\rho$ . Now we can consider  $\overline{R}_{\delta\gamma\beta}^\alpha$  in (2.19), as the components of the  $h$ -curvature tensor of  $\overline{M}$ . Therefore the norm of the  $h$ -curvature tensor  $R_{ijk}^h$  with respect to the metric tensor  $g$  is given by

$$\|R_{ijk}^h\|_g^2 = R_{ijk}^h R_h^{ijk} = \frac{1}{\rho'^4} \|\overline{R}_{\delta\gamma\beta}^\alpha - \rho''^2 (f_{\gamma\beta} \delta_\delta^\alpha - f_{\delta\beta} \delta_\gamma^\alpha)\|_f^2 + 4n \left(\frac{\rho'''}{\rho'}\right)^2.$$

By definition, in a critical point  $o$  at  $t = 0$  we have  $\rho'(0) = 0$ . From  $\rho''(0) = \phi(o)$  in (1.16) and the fact that  $f_{\gamma\beta}$ , the components of the metric tensor  $f$  and  $h$ -curvature tensor  $\overline{R}_{\delta\gamma\beta}^\alpha$  of  $\overline{M}$  are independent of  $t$ , we can conclude that the above equation, as  $t$  tends to zero, becomes

$$\overline{R}_{\delta\gamma\beta}^\alpha = \phi(o)^2 (f_{\gamma\beta} \delta_\delta^\alpha - f_{\delta\beta} \delta_\gamma^\alpha).$$

By means of Proposition 4, the  $t$ -levels of  $\rho$  are hyperspheres with center  $o$  and therefore  $\phi(o) \neq 0$ . Hence  $\overline{M}$  has positive constant sectional curvature  $\overline{K} = \phi(o)^2$ .  $\square$

As it is mentioned on the proof of the above lemma,  $\phi(o)$  does not vanish in this case and we have

$$\lim_{t \rightarrow 0} \frac{\rho'(t)}{t} = \rho''(0) = \phi(o) \neq 0. \quad (4.21)$$

Therefore  $\rho'(t)$  and  $t$  are of the same order.

In case there is a solution of (1.1) with two critical points, and the Finsler manifold  $(M, g)$  is compact, an extension of Milnor Theorem [16] implies that  $M$  is homeomorphic to an  $n$ -sphere. Thus we have the following proposition.

**Proposition 5.** *Let  $(M, g)$  ( $\dim M > 2$ ) be a simply connected and compact Finsler manifold which admits a solution  $\rho$  of (1.1) with two critical points, then  $M$  is homeomorphic to an  $n$ -sphere.*

## 5 A classification of complete Finsler manifolds.

Here we summarize the above results on the existence of solutions of Eq.(1.1).

**Theorem 2.** *Let  $(M, g)$  be a connected complete Finsler manifold of dimension  $n \geq 2$ . If  $M$  admits a non-trivial solution of  $\nabla^H \nabla^H \rho = \phi g$ , where  $\nabla^H$  is the Cartan horizontal covariant derivative, then depending on the number of critical points of  $\rho$ , i.e. zero, one or two respectively, it is conformal to*

- (a) *A direct product  $J \times \overline{M}$  of an open interval  $J$  of the real line and an  $(n - 1)$ -dimensional complete Finsler manifold  $\overline{M}$ .*
- (b) *An  $n$ -dimensional Euclidean space.*
- (c) *An  $n$ -dimensional unit sphere in an Euclidean space.*

*Proof.* Let  $\overline{M}_0$  be a  $t$ -level of  $\rho$  and  $\overline{M}$  an  $(n - 1)$ -dimensional Finsler manifold having the metric form  $\overline{ds}^2 = f_{\gamma\beta} du^\gamma du^\beta$  defined as in (1.17), which coincides with  $\overline{M}_0$  as a set of points. First of all we note that in a complete manifold  $M$ ,  $\overline{M}_0$  is complete with respect to the induced Finsler metric. In fact, the distance between points of  $\overline{M}_0$  with respect to the induced metric is not shorter than that of  $M$ , and hence a Cauchy sequence of points in  $\overline{M}_0$  is also a Cauchy sequence in  $M$ . Since  $M$  is complete and  $\overline{M}_0$  is closed in  $M$ , every Cauchy sequence has its limiting point in  $\overline{M}_0$ , hence  $\overline{M}_0$  is complete. By means of (1.17), the induced tensor metric of  $\overline{M}_0$  is proportional to the tensor metric of  $\overline{M}$ , so they are homothetic to each other. Therefore  $\overline{M}$  is also complete.

**case (a)** If one of the  $t$ -geodesics orthogonal to  $\overline{M}_0$  has no critical point, then non of them has. Since  $M$  is complete,  $t$ -geodesics are extendable to whole interval  $I = (-\infty, +\infty)$  of the arc-length  $t$ . So we can define the following map

$$\nu : I \times \overline{M} \rightarrow M,$$

$$(t, p) \rightarrow p(t),$$

where the point  $p(t)$  corresponding to the value  $t \in I$  lies on a  $t$ -level of  $\rho$  and we have  $\nu(0, \overline{M}) = \overline{M}_0$ . On the other hand,  $\overline{M}$  is diffeomorphic to the  $t$ -levels, therefore the map  $\nu$  is a diffeomorphism of  $I \times \overline{M}$  into  $M$ . Since  $M$  is connected, any point  $q$  of  $M$  is joined to a point of  $\overline{M}_0$  by a curve  $\mathcal{C}$ . By extending local diffeomorphism among  $t$ -levels of  $\rho$  through points of  $\mathcal{C}$ , we can see that  $q$  is an image of a point  $(t, p)$  of  $I \times \overline{M}$ . Thus the map  $\nu$  is a diffeomorphism of  $I \times \overline{M}$  onto  $M$ . Therefore from the proof of Lemma 2, the metric form of  $M$  is expressed as

$$ds^2 = dt^2 + (\rho')^2 \overline{ds}^2, \tag{5.22}$$

where  $\overline{ds}^2$  is the metric form of  $\overline{M}$ .

After a reparametrization of  $t$ -geodesics such that  $\rho' > 0$ , we define a parameter  $r$  by

$$r(t) = \int_0^t \frac{1}{\rho'} dt \quad t \in I. \quad (5.23)$$

$r(t)$  is an increasing monotone function of  $t$ . Let

$$r_1 = \lim_{t \rightarrow -\infty} r(t), \quad \text{and} \quad r_2 = \lim_{t \rightarrow +\infty} r(t),$$

where  $r_1$  and  $r_2$  may be infinite, and let  $J$  be the interval  $(r_1, r_2)$ . Now because  $dr = \frac{1}{\rho'} dt$ , we can write the metric form (5.22) as follows

$$ds^2 = (\rho')^2 (dr^2 + \overline{ds}^2), \quad r \in J. \quad (5.24)$$

Thus  $M$  is conformal to the direct product  $J \times \overline{M}$ .

**case (b)** If one of the  $t$ -geodesics issuing from points of  $\overline{M}_0$  has a critical point  $o$  at a distance  $t_0$  from  $\overline{M}_0$  and no critical point in the opposite direction, then all such curves have the same behavior, and  $o$  is the only critical point of  $\rho$ . Let's parameterize  $t$ -geodesics by arc-length  $t$  measured from  $o$  and put  $I = (0, \infty)$ . The map  $\nu : I \times \overline{M} \rightarrow M$  defined as in the case (a), is a diffeomorphism of  $I \times \overline{M}$  onto the open set  $M \setminus \{o\}$ . By Lemma 2,  $\overline{M}$  with the metric form  $\overline{ds}^2$  is a space of positive constant curvature  $(\phi(o))^2$ . If we suppose that  $\phi(o) > 0$  in  $I$  and put  $\bar{c} = \phi(o)$ , then we can define a parameter  $r$  for the  $t$ -geodesics by

$$r(t) = e^{\bar{c} \int_{t_0}^t \frac{dt}{\rho'}}, \quad t \in I, \quad (5.25)$$

which is an increasing monotone function of  $t$ . From (4.21), we see that  $\rho'(t)$  is of the same order as  $t$ , when  $t$  tends to zero, hence

$$\lim_{t \rightarrow 0} r(t) = e^{\bar{c}(\lim_{t \rightarrow 0} \int \frac{dt}{t})} = e^{\bar{c}(-\infty)} = 0.$$

If we put  $r_2 = \lim_{t \rightarrow \infty} r(t) \leq \infty$ , then the parameter  $r$  varies in the interval  $[0, r_2)$  as  $t$  varies in  $[0, +\infty)$ . Since

$$\frac{dr}{r} = \frac{\bar{c} dt}{\rho'(t)},$$

the metric form of  $M$  is equal to

$$ds^2 = \left( \frac{\rho'(t)}{r(t)\bar{c}} \right)^2 [dr^2 + r^2 \bar{c}^2 \overline{ds}^2], \quad 0 < r < r_2,$$

in  $M \setminus \{o\}$ . The expression in the brackets is the polar form of an  $n$ -dimensional Euclidean metric. On the other hand, taking into account orders of  $\rho'(t)$  and  $r(t)$ , we can see that the coefficient  $\frac{\rho'(t)}{r(t)}$  is not equal to zero but it is differentiable at  $o$ . Thus  $M$  is conformal to the Euclidean ball of radius  $r$ . Since  $r_2$  can be increased without bound,  $M$  is conformal to a flat space and by definition, it is conformally flat.

**case (c)** If one of the  $t$ -geodesics issuing from points of  $\overline{M}_0$  has two critical points  $o$  and  $o'$  in opposite directions at distances  $t_1$  and  $t_2$  respectively, then so do all such curves, and only the points  $o$  and  $o'$  are critical in  $M$ . We parameterize the  $t$ -geodesics by the arc-length  $t$  measured from  $o$ . Let the distance from  $o$  to  $o'$  be equal to  $2t_0$ , and put  $I = (0, 2t_0)$ . Moreover let  $\overline{M}_0$  be the  $t$ -level of  $\rho$  corresponding to  $t_0$ . The map  $v : I \times \overline{M} \longrightarrow M$ , defined as in the case (a), is a diffeomorphism of  $I \times \overline{M}$  onto the open set  $M \setminus \{o, o'\}$ . The metric form of  $M$  is written as in (5.22), where from the Lemma 2, the manifold  $\overline{M}$  with metric form  $\overline{ds}^2$  is a space of positive constant curvature  $\overline{k} = \phi(o)^2 = \phi(o')^2$ . Let  $\rho'(t) > 0$  on the interval  $I$ , that is  $\rho$  is an increasing function on  $I$ , so we have  $\rho''(0) > 0$  and  $\rho''(2t_0) < 0$ . Now we can define a parameter  $\theta$  by

$$\theta(t) = 2 \arctan \exp \overline{c} \int_{t_0}^t \frac{dt}{\rho'(t)}, \quad t \in I,$$

where  $\overline{c} = \phi(o) = -\phi(o')$ .  $\theta$  is an increasing monotone function of  $t$  and we have

$$\lim_{t \rightarrow 0} \theta(t) = 0, \quad \theta(t_0) = \frac{\pi}{2}, \quad \lim_{t \rightarrow 2t_0} \theta(t) = \pi.$$

Hence  $\theta$  varies in the closed interval  $[0, \pi]$  as  $t$  varies in  $[0, 2t_0]$ . Since we have

$$\frac{d\theta}{\sin \theta} = \frac{\overline{c} dt}{\rho'(t)},$$

the metric form of  $M$  is equal to

$$ds^2 = \left( \frac{\rho'(t)}{\overline{c} \sin \theta(t)} \right)^2 [d\theta^2 + (\sin \theta)^2 \overline{c}^2 \overline{ds}^2], \quad 0 < \theta < \pi, \quad (5.26)$$

in  $M \setminus \{o, o'\}$ . The expression in the brackets is the polar form of the Finsler metric on an  $n$ -sphere [18]. Taking into account orders of  $\rho'(u)$  and  $\sin \theta(u)$ , we can verify that the factor  $\rho'(u)/\sin \theta(u)$  is not equal to zero but it is differentiable at both the critical points  $o$  and  $o'$ . Therefore  $(M, g)$  is conformal to a Finsler metric on an  $n$ -sphere.  $\square$

## 6 Example of Finsler metrics with positive constant curvature.

Describing the Finsler metrics of constant flag curvature is one of the fundamental problems in Finsler geometry. Historically, the first set of non-Riemannian Finsler metrics of constant flag curvature are the Hilbert-Klein metric and the Funk metric on a strongly convex domain. In 1963, Funk [12] completely determined the local structure of two-dimensional projectively flat Finsler metrics with constant flag curvature. The Funk metric is positively complete and non-reversible with  $K = -\frac{1}{4}$  and the Hilbert-Klein metric is complete and reversible with  $K = -1$ . Both of them are locally projectively flat. Yasuda and Shimada [23] in 1977, classified Randers metrics of constant flag curvature, which has been rectified and completed in

a joint work of D. Bao, C. Robles and Z. Shen [4] in 2004. Akbar-Zadeh [2] in 1988, proved that a closed Finsler manifold with constant flag curvature  $K$  is locally Minkowskian if  $K = 0$ , and Riemannian if  $K = -1$ . In the case  $K = 1$ , Shen [19] asserts that the Finsler manifold must be diffeomorphic to sphere, provided that it is simply connected. Then, Bryant [7] showed that up to diffeomorphism, there is exactly a 2-parameter family of locally projectively flat Finsler metrics on  $S^2$  with  $K = 1$  and the only reversible one is the standard Riemannian metric. He has also extended his construction to higher dimensional spheres. These Bryant's examples are projectively flat and none of them is of Randers type. In 2000, Bao and Shen [5] constructed a family of non-projectively flat Finsler metrics on  $S^3$  with  $K = 1$ , using the Lie group structure of  $S^3$ . They also produced, for each constant  $K > 1$ , an explicit example of a compact boundaryless non-projectively flat Randers space with constant positive flag curvature  $K$ .

Here, based on a Finsler metric of positive constant curvature on certain hypersurfaces of the Finsler space  $(M, g)$ , we find some conditions for  $g$  to be a Finsler metric of positive constant curvature. This constructed Finsler metric is not necessarily of Randers type nor projectively flat. Without loss of generality, we have so far considered some hypersurfaces of  $\mathbb{R}^{n+1}$ .

**Proposition 6.** *Let  $\overline{M}$  be a regular hypersurface of  $\mathbb{R}^{n+1}$  defined by  $\rho = \text{constant}$ . The scalar function  $\rho$  is a solution of  $\nabla^H \nabla^H \rho = K^2 \rho g$  on the Finsler space  $(\mathbb{R}^{n+1}, g)$  and  $K$  is a constant number. If there exists a Finsler metric  $\overline{g}$  defined on  $\overline{M}$  with positive constant curvature, then  $g$  is a Finsler metric of this kind on  $\mathbb{R}^{n+1}$ .*

*Proof.* Let  $\rho$  be a differentiable scalar function on the Finsler space  $(\mathbb{R}^{n+1}, g_{ij})$  satisfying

$$\nabla_i^H \nabla_j^H \rho = K^2 \rho g_{ij}, \quad (6.27)$$

where  $\nabla^H$  denotes the Cartan  $h$ -covariant derivative,  $K$  is a positive constant,  $(u^i, v^i)$  is the local coordinate system on  $T\mathbb{R}^{n+1}$  and  $i, j$  run over the range  $1, \dots, n+1$ . Along the geodesics of  $(\mathbb{R}^{n+1}, g)$  with arc-length parameter  $t = u^{n+1}$ , equation (6.27) reduces to  $\frac{d^2 \rho}{(dt)^2} = K^2 \rho$ . The general solution of this differential equation is given by  $\rho(t) = a \cos(Kt) + b$ , where  $a$  and  $b$  are constants. Differentiating with respect to  $t$  gives  $\rho'(t) = -aK \sin(Kt)$ . Therefore, there are two critical points corresponding to  $t = 0$  and  $t = \frac{\pi}{K}$ . Let  $\overline{M}$  be the hypersurface of  $\mathbb{R}^{n+1}$  defined by  $\rho = \text{constant}$  and  $\overline{g}$  a Finsler metric with positive constant curvature  $\overline{K}$  on  $\overline{M}$ . If we put  $a = \frac{1}{\overline{K}}$ , then from (5.26) the Finsler structure of  $\mathbb{R}^{n+1}$  becomes

$$ds^2 := g_{ij} du^i du^j = (dt)^2 + (\sin(Kt))^2 \overline{K}^2 \overline{g}_{\alpha\beta} du^\alpha du^\beta, \quad (6.28)$$

where  $\alpha, \beta = 1, \dots, n$ . This is the polar form of a Finsler metric on an  $n$ -sphere in  $\mathbb{R}^{n+1}$  with positive constant curvature  $K$  [18].  $\square$

The same method can be applied to construct a Finsler metric with positive constant curvature on an  $(n+1)$ -dimensional complete Finsler manifold  $(M, g)$ , starting from a Finsler metric of this kind on a hypersurface of  $M$ .

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